**Homework #6. Due Thursday, March 5th, in class**

**Reading:**

1. For this assignment: 3.1 + class notes (Lectures 12-13).
2. For next week's classes: 3.3 (continuity). I will be following the exposition in the book for this section quite closely.

**Mandatory reading:** Read the part of 3.2 dealing with one-sided limits (up to and including Theorem 3.14) before the class on Tue, March 3.

**Problems:**

**Problem 1:** In each part of the problem prove that $\lim\limits_{x\to a} f(x) = L$ directly using $\varepsilon$-$\delta$ definition of limit and give an explicit formula for $\delta$ in terms of $\varepsilon$.

(a) $f(x) = x^3$, $a = 2$, $L = 8$.

(b) $f(x) = \frac{1}{x}$, $a = 1$, $L = 1$. **Hint:** First find (explicit) $\delta_0 > 0$ and $C > 0$ such that inequality $|x - 1| < \delta_0$ implies $|x| \geq C$.

(c) $f(x) = x^2$, $a \in \mathbb{R}$ is arbitrary, $L = a^2$ (naturally, your answer in this part will depend on $a$).

**Problem 2:** Let $f(x) = x^2$.

(a) Fix $c, d \in \mathbb{R}$ with $c < d$. Use your answer in 1(c) to prove that for every $\varepsilon > 0$ there is $\delta > 0$ s.t. $|x^2 - a^2| < \varepsilon$ for all $x, a \in [c, d]$ s.t. $|x - a| < \delta$ (note that $\delta$ here is allowed to depend on $\varepsilon$ but not on $x$ and $a$). **Hint:** Even though your answer in 1(c) depends on $a$, keep in mind that if some $\delta$ works (for a given $a$ and $\varepsilon$), then any smaller $\delta$ also works.

(b) Now fix an arbitrary $\varepsilon > 0$. Prove that there is no $\delta > 0$ s.t. $|x^2 - a^2| < \varepsilon$ for all $x, a \in \mathbb{R}$ s.t. $|x - a| < \delta$. **Hint:** Assume that such $\delta$ exists. Set $x = a + \frac{\delta}{2}$ and let $a$ go to $+\infty$ to reach a contradiction.

**Problem 3:**

(a) Let $\{a_n\}$ be a sequence and $L$ a real number. Prove that $\lim\limits_{n\to\infty} a_n = L$ $\iff$ $\lim\limits_{n\to\infty} |a_n - L| = 0$. **Hint:** This is almost immediate from the definition of limit.

(b) Now suppose we are given two sequences $\{a_n\}, \{b_n\}$ and let $L \in \mathbb{R}$. Suppose that $|a_n - L| \leq b_n$ for all $n$ and $\lim\limits_{n\to\infty} b_n = 0$. Prove that $\lim\limits_{n\to\infty} a_n = L$. **Hint:** use (a) and the squeeze theorem.

(c) State and prove the analogues of (a) and (b) for limits of functions.

**Note:** Recall that we used the result of (b) in the proof of Bolzano-Weierstrass theorem. The result of this problem will be frequently used later in the course.

**Problem 4:** Let $a \in \mathbb{R}$.

(a) Let $f$ be a real function which is defined near $a$ and converges at $a$, and let $L = \lim_{x \to a} f(x)$. Use the $\varepsilon$-$\delta$ definition of limit to prove that the function $|f|$ also converges at $a$ and $\lim_{x \to a} |f(x)| = |L|$.

(b) Let $u, v \in \mathbb{R}$. Prove that $\max\{u, v\} = \frac{1}{2}(u + v + |u - v|)$ and $\min\{u, v\} = \frac{1}{2}(u + v - |u - v|)$.

(c) Now let $f$ and $g$ be real functions which are defined near $a$ and converge at $a$, and let $L = \lim_{x \to a} f(x)$ and $M = \lim_{x \to a} g(x)$. Prove that the functions $\max\{f, g\}$ and $\min\{f, g\}$ are also defined near $a$ and converge at $a$ and that $\lim_{x \to a} \max\{f(x), g(x)\} = \max\{L, M\}$ and $\lim_{x \to a} \min\{f(x), g(x)\} = \min\{L, M\}$. **Hint:** Use (a),(b) and a suitable theorem from class. **Warning:** you cannot assume that $f(x) \leq g(x)$ for all $x$ near $a$ or $f(x) \geq g(x)$ for all $x$ near $a$.

**Problem 5:** Problem 3 from midterm 1. **Hint:** As discussed in class, the problem reduces to showing that $\sup(A)$ is an upper bound for $S$ and that $\sup(S)$ is an upper bound for $A$. However, you should present a complete solution (explaining why the above statement is sufficient to solve the problem). You do not need to hand in solution to this problem if you received the score of 7 or 8 on the test.

**Problem 6:** Problem 4 from midterm 1. **Hint:** Use (a) to solve (b). If you do things in a right way, algebra should be very simple. In (a) $C$ and $N$ should be explicit numbers (like 25) – the answer should not involve any unknowns.

**Problem 7:** Problem 5 from midterm 1. **Note:** You can use any theorem from class – I did not mean to ask you to reprove any of the theorems. **Hint:** For the difficult part of (b) (if (i) is false, then (ii) is true), the first thing you need to do is to (correctly) write the negation of the condition in (a). To proceed with the proof use the fact that a subsequence of a subsequence is a subsequence of the original sequence (do you remember where we used this in class?)

Before the next problem we generalize the notion of limit for functions defined on metric spaces. Let $(X, d)$ be a metric space. Recall that given a point $a \in X$ and $\delta > 0$, we defined the open ball of radius $\delta$ centered at $a$ to be the set $B_\delta(a) = \{x \in X : d(x, a) < \delta\}$ (thus, in the case when $X = \mathbb{R}$ with the standard metric, $B_\delta(a)$ is just the open interval $(a - \delta, a + \delta)$).

The set $B^\circ_\delta(a) = B_\delta(a) \setminus \{a\}$ is called the punctured open ball of radius $\delta$ centered at $a$ (the superscript $\circ$ represents the puncture).

Now fix $a \in \mathbb{R}$, and assume that the punctured open ball $B^\circ_\delta(a)$ is non-empty for every $a \in X$ and $\delta > 0$ (note that there are metric spaces where this property fails for every $a$, e.g. the discrete metric space introduced in HW#1.6). Let $E$ be a subset of $X$, and assume that $E$ contains $B^\circ_{\delta_0}(a)$ for some $\delta_0 > 0$. Let $f : E \to \mathbb{R}$ be a function. We write $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ for all } x \in B^\circ_\delta(a).$$

**Problem 8 (bonus):** The goal of this problem is to show that the definition of limit of a sequence is a special case of the definition of limit in metric spaces.

(a) Let $X = \mathbb{N} \cup \{\infty\}$ (we treat $\infty$ as a formal symbol). Define the function $d : X \times X \to \mathbb{R}$ by setting

$$d(n,m) = \begin{cases} \left|\frac{1}{n} - \frac{1}{m}\right| & \text{if } n, m \in \mathbb{N} \\ \frac{1}{n} & \text{if } n \in \mathbb{N}, m = \infty \\ \frac{1}{m} & \text{if } m \in \mathbb{N}, n = \infty \\ 0 & \text{if } n = m = \infty \end{cases}$$

In other words, we can say that $d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$ for all $n, m \in X$ if we adopt the convention that $\frac{1}{\infty} = 0$.

Prove that $(X, d)$ is a metric space.

(b) Now recall that a sequence $\{f_n\}$ of real numbers is simply a function $f : \mathbb{N} \to \mathbb{R}$ (where $f(n) = f_n$). Thus, we can think of $f$ as a function defined on any punctured open ball $B^\circ_\delta(\infty)$ centered at $\infty$ in the above metric space $(X, d)$. Prove that if $L$ is some real number, then $\lim_{n\to\infty} f_n = L$ (in the usual sense) $\iff$ $\lim_{n\to\infty} f(n) = L$ as a limit in the metric space $(X, d)$ defined in (a).