Homework #5. Due Thursday, February 26th, in class

Reading:

1. For this assignment: 1.6 and 2.4 + class notes (Lectures 10-11).

2. For next week's classes: 3.1 (limits of functions) and beginning of 3.3 (continuity); 3.2 will be assigned as required reading.

Note: In Lecture 11 we proved that various sets are countable by constructing enumerations for those sets. In homework problems below this will usually not be necessary – you can prove countability by combining various results discussed in class.

Problems:

Problem 1: Let A and B be sets.

(a) Suppose that A is countable and there exists a bijection $\psi : A \to B$. Prove that B is also countable (this follows almost immediately from the definition).

(b) Now suppose that A is countable, B is infinite and there exists a surjection $\phi : A \to B$. Prove that B is countable. **Hint:** Show that there is a subset C of A such that ϕ restricted to C is a bijection from C to B and then use (a) and a result from class.

Problem 2: Let $A_1, \ldots, A_n$ be a finite collection of sets. Define the Cartesian product $A_1 \times \ldots \times A_n$ to be the set of all n-tuples $(a_1, \ldots, a_n)$ with $a_i \in A_i$ for each i. Prove that if each A_i is countable, then $A_1 \times \ldots \times A_n$ is countable. Note that for $n = 2$ this holds by Lemma 11.5 from class. **Hint:** Use induction on n. To complete the induction step show that there is a natural bijection between $A_1 \times \ldots \times A_n \times A_{n+1}$ and $(A_1 \times \ldots \times A_n) \times A_{n+1}$ (where the set on the right by definition is the Cartesian product of two sets, $A_1 \times \ldots \times A_n$ and A_{n+1}).

Problem 3: A number α is called algebraic if α is a root of a (nonzero) polynomial with integers coefficients, that is, if there exist integers $c_0, \ldots, c_n$, not all 0 such that $\sum\limits_{k=0}^{n} c_k \alpha^k = 0$. Note that all rational numbers are algebraic (if $\alpha = \frac{p}{q}$, then $q\alpha - p = 0$), but many irrational numbers are algebraic as well (e.g. $\sqrt{2}$ is algebraic as $(\sqrt{2})^2 - 2 = 0$). The goal of this problem is to prove that the set of all algebraic numbers is countable.

(a) For a fixed integer $n \geq 0$, let Z_n be the set of all polynomials of degree at most n with integer coefficients. Prove that each Z_n is countable. **Hint:** Use Problem 2 to construct a countable set Y_n and a bijection $\phi_n : Y_n \to Z_n$.

(b) Use (a) and the result from class to show that the set of all polynomials with integer coefficients (of arbitrary degree) is countable.

(c) Finally use (b) and the fact that every polynomial has finitely many roots to show that the set of all algebraic numbers is countable.

Problem 4: Let x be a real number. Prove that there exists a sequence $\{q_n\}$ of rational numbers which converges to x. **Hint:** Start by constructing two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers such that $a_n < x < b_n$ for all n and $\lim a_n = \lim b_n = x$. Once you have such sequences, think which theorems can possibly be applicable.

Problem 5: Consider the sequence $\{a_n\}$ defined by $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ for all $n \geq 1$. Recall that in Lecture 9 we proved that this sequence is convergent by applying the monotone convergence theorem. The goal of this problem is prove convergence of $\{a_n\}$ by a different method, namely by showing that it is Cauchy (and hence also convergent by Theorem 10.2). In Lecture 10 we proved that $\{a_n\}$ is Cauchy if there is $\alpha \in (0, 1)$ such that

$$|a_{n+2} - a_{n+1}| \leq \alpha |a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}. \qquad (***)$$

Use the inequality $a_n \geq \sqrt{2}$ established in Lecture 9 to show that (***) holds in this case for suitable α (which you need to find); note that α cannot depend on n.

Before the next two problems we define the notions of convergent and Cauchy sequences in arbitrary metric spaces. Recall that the notion of a metric space was introduced in Homework#1 and some examples of metric spaces were discussed in Homeworks#1 and #2.

So let (X, d) be a metric space, $\{a_n\}$ a sequence in X (that is, a sequences all of whose elements lie in X) and L some point of X.

(i) We say that $\{a_n\}$ converges to L and write $\lim a_n = L$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a_n, L) < \varepsilon$ for all $n \geq N$.

(ii) We say that $\{a_n\}$ is Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ for all $n, m \geq N$.

As you can see, this is really the same definition as for real sequences except that $|x - y|$ is replaced $d(x, y)$.

One may ask which limit theorems we established for $\mathbb{R}$ extend to arbitrary metric spaces. Some results (e.g. sum rule, product rule or comparison theorem) do not even make sense as statements since in general one cannot talk about sums/products or inequalities between elements of X, while some others remain true with essentially the same proofs (recall that triangle inequality in the general metric space has the form $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in \mathbb{R}$).

Problem 6: Let (X, d) be a metric space. Prove that any convergent sequence in X is Cauchy. **Hint:** The proof is virtually identical to the corresponding result from class.

Problem 7: Again let (X, d) be a metric space.

(a) (optional) Assume that (X, d) has the property that every Cauchy sequence has a convergent subsequence. Prove that every Cauchy sequence in X is convergent.

(b) One can consider rational numbers $\mathbb{Q}$ as a metric space with the standard metric $d(x, y) = |x - y|$. Prove that there exist Cauchy sequences in $(\mathbb{Q}, d)$ which do NOT converge (to an element of $\mathbb{Q}$). **Hint:** use the result of one of the earlier problems in this homework.