Homework #4. Due Thursday, February 12th, in class

Reading:

1. For this assignment: Sections 2.1, 2.2, first part of 2.3 + class notes (Lectures 7-8).

2. For next week's classes: the second part of 2.3 (Bolzano-Weierstrass theorem), 2.4 (Cauchy sequences) and 1.6 (countable and uncountable sets).

Mandatory reading 1: Before the class on Tuesday, Feb 10 read the proof of Bolzano-Weierstrass theorem (Theorem 2.26). I will present the proof in class on Tuesday ASSUMING that you have already read it in the book.

Mandatory reading 2: There are a lot of important results in Chapter 2, and we did not discuss some of them in class due to time constraints. The following is the list of such items which are left as required reading (all of them will be used later in the course):

(a) squeeze theorem (Theorem 2.9). Note that squeeze theorem has two parts. The first part is probably familiar to you from calculus, but the second part is likely new for many of you.
(b) Definition 2.14 and Theorem 2.15. Definition and basic properties of infinite limits.
(c) Examples 2.20 and 2.21. Additional applications of the monotone convergence theorem.

Problems:

Problem 1: Let $\{a_n\}$ be a sequence which has finitely many distinct terms (e.g. $1, 3, 2, 3, 2, 3, 2, \ldots$). Prove that if $\{a_n\}$ converges, then there exists $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $a_n = C$ for all $n \geq N$ (such sequences are called eventually constant).

Problem 2: Let $\{a_n\}$ be a sequence and L a real number. Prove that $\lim\limits_{n\to\infty} a_n = L \iff$ for every $k \in \mathbb{N}$ there exists $N = N(k)$ such that $|a_n - L| < \frac{1}{k}$ for all $n \geq N$ (the point is that instead of verifying the inequality in the definition of limit for all $\varepsilon > 0$, it suffices to check that condition for ε of the form $\frac{1}{k}$ with $k \in \mathbb{N}$). **Hint:** One direction is immediate, and the other direction follows from one of the problems in earlier homeworks.

Problem 3: Let $\{a_n\}$ be a sequence with $a_n > 0$ for all n. Prove that $a_n \to +\infty$ (see Definition 2.14) $\iff \frac{1}{a_n} \to 0$.

Problem 4: Let $\{a_n\}$ be a sequence. Prove that $\{a_n\}$ is unbounded above $\iff$ there is a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = +\infty$.

Hint: for the forward direction ($\Rightarrow$) show that one can construct a sequence of natural numbers $n_1 < n_2 < \ldots$ such that $a_{n_k} > k$ for all $k \in \mathbb{N}$. Such a sequence can be constructed inductively: suppose that for some $m \geq 1$ we have already constructed $n_1 < n_2 < \ldots < n_m$ such that $a_{n_k} > k$ for $k = 1, \ldots, m$. Assume that the process cannot be continued, that is, one cannot choose $n_{m+1} \in \mathbb{N}$ such that $n_{m+1} > n_m$ and $a_{n_{m+1}} > m + 1$, and deduce that the sequence $\{a_n\}$ must be bounded, reaching a contradiction.

Problem 5: The goal of this problem is to prove the "quotient rule" for limits (the last part of Theorem 2.12 from the book or Theorem 7.4 from class). Note that it suffices to prove that $\lim \frac{1}{b_n} = \frac{1}{\lim b_n}$ (where $b_n \neq 0$ for all n and $\lim b_n \neq 0$) – once this is done, the general case follows from the product rule.

So, let $\{b_n\}$ be a convergent sequence such that $b_n \neq 0$ for all n and $\lim b_n \neq 0$. Let $L = \lim b_n$.

(a) Show that there exists $N \in \mathbb{N}$ such that $|b_n| > \frac{|L|}{2}$ for all $n \geq N$. It may be convenient to consider the cases $L > 0$ and $L < 0$ separately.

(b) Use (a) to prove that the sequence $\left\{\frac{1}{b_n}\right\}$ is bounded.

(c) Now use (b) to prove that $\left\{\frac{1}{b_n}\right\}$ converges and $\lim \frac{1}{b_n} = \frac{1}{L}$

Problem 6: Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1+x}{2}$. Fix some $x_0 \in \mathbb{R}$, and define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$.

(a) Prove that if $1 \leq x$, then $1 \leq f(x) \leq x$. Also prove that if $x \leq 1$, then $x \leq f(x) \leq 1$.

(b) Use (a) and the monotone convergence theorem to prove that the sequence $\{x_n\}$ converges to 1 (regardless of the value of x_0).

Problem 7: Problem 2.2.9 from Wade's book.