Homework #3. Due Thursday, February 5th, in class

Reading:

1. For this assignment: Sections 1.4, 1.5, 2.1 + class notes (Lectures 5-6).

2. For next week's classes: the second part of 2.1 (starting with the definition of subsequence), 2.2 and first part of 2.3. I will not be talking about infinite limits (part of 2.2 between Definition 2.14 and Corollary 2.16) in class – this part is a required reading, but it is not necessary to read this material before next week's classes.

Problems:

Problem 1: Let A and B be non-empty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$. Prove that A is bounded above, B is bounded below and $\sup(A) \leq \inf(B)$. **Hint:** This can proved directly from the definitions of supremum and infimum without any computations or using any theorems, but you need to proceed in two steps.

Problem 2: Use induction to prove the formula for the sum of a (finite) geometric progression: $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$.

Problem 3: Prove the following inequalities by induction:

(i) $n < 2^n$ for all $n \in \mathbb{N}$

(ii) $n^2 < 2^n$ for all integers $n \geq 5$

Problem 4: Use induction to prove Bernoulli's inequality:

$$(1 + x)^n \geq 1 + nx \text{ for all } n \in \mathbb{N} \text{ and } x \geq -1.$$

Clearly indicate where you use that $x \geq -1$ in your argument

Before the next problem we make a small remark about the definition of the limit of a sequence. According to the definition from class, a sequence $\{a_n\}$ converges to some $L \in \mathbb{R}$ if

$$\text{for every } \varepsilon > 0 \text{ there is } N(\varepsilon) \in \mathbb{N} \text{ s.t. } |a_n - L| < \varepsilon \text{ for all } n \geq N(\varepsilon) \quad (1).$$

We claim that this condition is equivalent to

$$\text{for every } \varepsilon > 0 \text{ there is } M(\varepsilon) \in \mathbb{R} \text{ s.t. } |a_n - L| < \varepsilon \text{ for all } n > M(\varepsilon) \quad (1').$$

(The difference is that in (1') $M(\varepsilon)$ is only required to be a real number, not necessarily a natural number, and non-strict inequality $n \geq N(\varepsilon)$ is replaced by the strict inequality $n > M(\varepsilon)$). Indeed, (1) clearly implies (1').

Conversely, suppose (1') holds. By the Archimedean property we can find $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > M(\varepsilon)$, and then $N(\varepsilon)$ clearly satisfies (1).

Problem 5: In each of the examples below prove that a sequence $\{a_n\}$ converges to L, and explicitly find a function $M(\varepsilon)$ satisfying (1') above (note that because we are using (1') instead of (1), there is no need to worry about taking the integer part, as we did in class).

(i) $a_n = \frac{2n^2+3}{n^2-n-\cos(n)}$, $L = 2$
(ii) $a_n = \frac{n}{4^n}$, $L = 0$.

Problem 6: Let $\{a_n\}$ and $\{b_n\}$ be sequences. Suppose that for every $\varepsilon > 0$ the following is true: $|a_n - 3| < \varepsilon$ for all $n > \frac{10}{\varepsilon^2}$ and $|b_n - 4| < \varepsilon$ for all $n > \frac{1}{\varepsilon^3}$. Find an explicit function $M(\varepsilon)$ such that $|a_n + b_n - 7| < \varepsilon$ for all $n > M(\varepsilon)$. **Hint:** The answer to this problem can be deduced from the proof of the theorem asserting that the limit of the sum of two convergent sequences is equal to the sum of their limits (see Theorem 2.12(i)).

Problem 7: Let $\{a_n\}$ be a sequence, and define b_k and c_k (with $k \in \mathbb{N}$) by $b_k = a_{2k-1}$ and $c_k = a_{2k}$, that is, $\{b_k\}$ and $\{c_k\}$ are subsequences of $\{a_n\}$ consisting of its elements located in odd (respectively even) positions. Suppose that $\{b_k\}$ and $\{c_k\}$ both converge and $\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = L$ for some $L \in \mathbb{R}$. Prove that $\{a_n\}$ converges to L as well. **Warning:** You cannot take for granted that $\{a_n\}$ converges to some real number.

Problem 8: Let $f : X \to Y$ be a function. Prove that the following conditions are equivalent:

(a) f is injective
(b) $f(A \cap C) = f(A) \cap f(C)$ for any two subsets A, C of X.

(In other words, prove that (a) implies (b) and (b) implies (a)).