Homework #2. Due Thursday, January 29th, in class

Reading:

1. For this assignment: Section 1.3 + class notes (Lectures 3-4).

2. For next week's classes: Section 1.5 and 2.1. We may briefly talk about induction as well (1.4), but I have not decided yet.

Problems:

Problem 1: Prove Claim 3.3 from class: let S be a non-empty subset of $\mathbb{R}$. Then $\max(S)$ exists (that is, S has a maximal element) $\iff$ S is bounded above and $\sup(S) \in S$.

Problem 2: Let S be a non-empty subset of $\mathbb{R}$. Let $UB(S)$ be the set of all upper bounds of S (note that this set may be empty) and $LB(S)$ be the set of all lower bounds of S. Also let $-S = \{-s : s \in S\}$

(i) Let $M \in \mathbb{R}$. Prove that $M = \sup(S)$ if and only if $M = \min(UB(S))$ (the minimal element of $UB(S)$). Also prove that $M = \inf(S)$ if and only if $M = \max(LB(S))$ (the maximal element of $LB(S)$). This is essentially a reformulation of the definition of sup and inf.

(ii) Let $y \in \mathbb{R}$. Prove that $y \in UB(S) \iff -y \in LB(-S)$.

(iii) Deduce from (ii) that $UB(S)$ has a minimum $\iff$ $LB(-S)$ has a maximum, and if they exist, then $\min(UB(S)) = -\max(LB(-S))$.

(iv) (practice) Combine (i)-(iii) to deduce the reflection principle as formulated in Lecture 4.

Problem 3: Use the Archimedean property to prove that for every real number $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Problem 4: Prove the following result, which can be thought of as a converse of the Approximation Theorem (Theorem 3.2). Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Let $M \in \mathbb{R}$ be an upper bound for S, and suppose that for all $\varepsilon > 0$ there exists $x \in S$ such that $M - \varepsilon < x \leq M$. Prove that $M = \sup(S)$.

Problem 5: Let A and B be non-empty bounded above subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$. Prove that $A + B$ is also bounded above and $\sup(A + B) = \sup(A) + \sup(B)$.

Problem 6: This problem introduces the notions of open and closed subsets of $\mathbb{R}$. Let S be a subset $\mathbb{R}$. We say that S is *open* if for every $x \in S$ there exists $\varepsilon > 0$ (which may depend on x) such that $(x - \varepsilon, x + \varepsilon) \subseteq S$ (thus, for every point of S there is some open interval centered at that point which is entirely contained in S). We say that S is closed if its complement $\mathbb{R} \setminus S$ is open.

(a) Prove that if S is an open interval (that is, $S = (a, b) = \{x \in \mathbb{R} : a < x < b\}$ for some $a < b$), then S is an open subset of $\mathbb{R}$. **Hint:** This is merely a reformulation of one of the results in HW#1.

(b) Prove that if S is a closed interval ($S = [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a \leq b$), then S is a closed subset of $\mathbb{R}$.

Before the next problem we define the notion of an open ball in any metric space (recall that the notion of a metric space was introduced in HW#1.6). Let (X, d) be a metric space. Given $x \in X$ and a real number $\varepsilon > 0$, we define $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$, the set of all points of X whose distance from x is less than ε. The set $B_\varepsilon(x)$ is called the *open ball of radius ε around x (or centered at x).*

Problem 7:

(a) Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Recall that (X, d) is a metric space by HW#1.6(b). Prove that $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for all $x \in X$ and $\varepsilon > 0$ (thus, an open ball of radius ε centered at x in this case is simply the open interval of length 2ε centered at x). **Hint:** The result follows directly from basic properties of absolute values.

(b) Now let $X = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, and define functions $d : X \times X \to \mathbb{R}$ and $D : X \times X \to \mathbb{R}$ by setting $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $D((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Note that $d((x_1, y_1), (x_2, y_2))$ is simply the distance (in the usual sense) between points (x_1, y_1) and (x_2, y_2) on the (Euclidean) plane $\mathbb{R}^2$. In this problem you can assume without proof that the pairs (X, d) and (X, D) are both metric spaces. The function d is called the *Euclidean metric* on $\mathbb{R}^2$ (for the above reason), and the function D is called the *Manhattan metric* on $\mathbb{R}^2$ (do you see why it is called this way?).

Now the actual problem: Describe the open ball $B_\varepsilon((x, y))$ in each of these two metric spaces (in both cases the answer is a simple geometric figure) (an answer + a brief explanation is sufficient).

Problem 8 (bonus): Now we define the notion of an open set in an arbitrary metric space. Let (X, d) be a metric space. A subset S of X is called open if for every $x \in S$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. Prove that if S is any open ball in X (that is, $S = B_\alpha(y)$ for some $y \in X$ and $\alpha > 0$), then S is an open subset of X.