Homework #11. Due Thursday, April 23rd, in class

Reading:

1. For this assignment: 5.2, 5.3 + class notes (Lectures 22-23).
2. For next week's classes: go over 6.1-6.3.

Problems:

Problem 1: Problem 4(b) from Midterm 2. If you received full credit (3 points) for one or both parts of this problem, you do not need to submit solution to the corresponding part(s).

Problem 2: Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and suppose that f has only finitely many points of discontinuity. Prove that f is integrable on $[a, b]$. **Hint:** Use induction on the number n of points of discontinuity. The base case ($n = 1$) was proved in Lecture 22.

Problem 3:

(a) Let $h : [a, b] \to \mathbb{R}$ be a function, and suppose that there are only finitely many points $x \in [a, b]$ such that $h(x) \neq 0$. Prove that h is integrable on $[a, b]$ and $\int_a^b h(x)\,dx = 0$. **Hint:** By Problem 1, to prove that h is integrable, it suffices to show that it is bounded. Then show that $U(P, h) \geq 0$ and $L(P, h) \leq 0$ for any partition P – combined with the fact that h is integrable, these inequalities imply that $\int_a^b h(x)\,dx = 0$ (why?)

In parts (b) and (c) below we assume that $f, g : [a, b] \to \mathbb{R}$ are bounded functions such that there are only finitely many points $x \in [a, b]$ with $f(x) \neq g(x)$.

(b) Prove that

$$\overline{\int_a^b} f(x)dx = \overline{\int_a^b} g(x)dx \quad \text{and} \quad \underline{\int_a^b} f(x)dx = \underline{\int_a^b} g(x)dx.$$

Hint: This follows from (a) (applied to suitable h) and HW#10.4.

(c) Deduce from (b) that if f is integrable on $[a, b]$, then g is also integrable on $[a, b]$.

Problem 4:

(a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function, let $m = \inf\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$. Prove that

$$\overline{\int_a^b} f(x)dx \leq M(b-a) \text{ and } \underline{\int_a^b} f(x)dx \geq m(b-a).$$

(b) Let $MD : \mathbb{R} \to \mathbb{R}$ be the modified Dirichlet function (see Lecture 15). Prove that for any real numbers $a < b$, the function MD is integrable on $[a,b]$ and $\int_a^b MD(x)dx = 0$. **Hint:** Fix a and b. Use (a) to show that for any $\varepsilon > 0$ there exists a function $f_\varepsilon : [a,b] \to \mathbb{R}$ such that

(i) there are only finitely many points $x \in [a,b]$ such that $f_\varepsilon(x) \neq MD(x)$.

(ii) $\sup\{f_\varepsilon(x) : x \in [a,b]\} < \varepsilon$ and $\inf\{f_\varepsilon(x) : x \in [a,b]\} = 0$.

Then use (a) (applied to f_ε) and 3(b) to finish the proof (you would need to use this argument not for a fixed ε, but for an arbitrary ε and then let ε tend to 0).

Problem 5: Prove the following generalization of FTC. Let $f : [a,b] \to \mathbb{R}$ be a function, and suppose f is discontinuous at only finitely many points. This implies that f is integrable on $[a,b]$ by Problem 2, hence also integrable on $[a,x]$ for every $a \leq x \leq b$ by Theorem 5.20 from the book. Define $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)dt$. Note that F is continuous on $[a,b]$ by Theorem 23.2 from class (=Theorem 5.26 from the book). Prove that F is differentiable at every point of (a,b) where f is continuous. **Hint:** First show that since f is only discontinuous at finitely many points, for any $c \in (a,b)$ s.t. f is continuous at c we can find $\delta > 0$ s.t. f is continuous on $[c-\delta, c+\delta]$. One can now prove that F is differentiable at c by applying FTC on the interval $[c-\delta, c+\delta]$ (If we define $F_c : [c-\delta, c+\delta] \to \mathbb{R}$ by $F_c(x) = \int_c^x f(t)dt$, how is F_c related to F?)

Problem 6: This problem outlines one of the possible ways to rigorously define the natural logarithm function (which we denote by $\log(x)$). Define the function $L : (0,+\infty) \to \mathbb{R}$ by $L(x) = \int_1^x \frac{1}{t}dt$. (By convention, if $a > b$ and f is integrable on $[b,a]$, we define $\int_a^b f(t)dt = -\int_b^a f(t)dt$). Of course, $L(x) = \log(x)$ for all x, but in this problem we use the notation L to emphasize that we are constructing a function 'from scratch' without assuming any of its properties for granted.

(a) Prove that L is differentiable and increasing on $(0,+\infty)$ and that $\lim_{x\to+\infty} L(x) = +\infty$. **Hint:** Once you prove that L is increasing, to prove that $\lim_{x\to+\infty} L(x) = +\infty$, it suffices to show that L is unbounded from above. To prove that latter, use lower sum estimate to show that $\int\limits_{2^n}^{2^{n+1}} \frac{1}{t}dt \geq \frac{1}{2}$ for every $n \in \mathbb{N}$ (we will use essentially the same trick in Lecture 24 to prove that the harmonic series diverges).

(b) Prove that $L(xy) = L(x) + L(y)$ for all $x, y > 0$. **Hint:** treat y as a constant and use FTC to show that the function $L(xy) - L(x)$ is constant as a function of x.

(c) (bonus) We can rigorously define the number e by the condition $L(e) = 1$ (the fact that there exists unique number with this property follows from (a) and IVT). Show that L is the inverse of the exponential function $x \mapsto e^x$, that is $L(e^x) = x$ for all $x \in \mathbb{R}$ and $e^{L(x)} = x$ for all $x > 0$. **Hint:** first show that the functions $A(x) = L(e^x)$ and $B(x) = e^{L(x)}$ satisfy $A(x + y) = A(x) + A(y)$ and $B(x + y) = B(x) + B(y)$ and use the result of HW#9.2 (the result of that problem remains true if instead of functions defined on the entire $\mathbb{R}$ we only consider functions defined on some open interval).