**Homework #10. Due Thursday, April 16th, in class**

**Reading:**

1. For this assignment: 4.4, 5.1 + class notes (Lectures 19-21).

2. For next week's classes: 5.1 (parts which were not covered in class), 5.2 and beginning of 5.3 (Theorem 5.28 and Example 5.29)

**Problems:**

**Problem 1:** As in the previous assignment, log denotes the natural logarithm (logarithm to the base $e$). Let $f(x) = \log(1+x)$ (the domain of $f$ is $(-1,\infty)$).

(a) Find the formula for $f^{(n)}(x)$ for every $n \in \mathbb{N}$. **Hint:** compute $f^{(n)}(x)$ for a few small values of $n$, use your computations to guess the formula (for arbitrary $n$) and then prove the formula rigorously by induction.

(b) Explicitly compute $P_{n,0}(x)$, the $n^{\text{th}}$ Taylor polynomial of $f$ centered at 0.

(c) Use Taylor's theorem to show that $|\log(\frac{1}{2}) - P_{7,0}(\frac{-1}{2})| < \frac{1}{1000}$.

**Problem 2:** Problem 3(b)(c) from Midterm#2. If you received full credit (5 points) for one or both parts of this problem, you do not need to submit solution to the corresponding part(s).

**Problem 3:** For each $n \in \mathbb{N}$ let $P_n$ be the partition of the interval $[0,1]$ into $n$ subintervals of the same length $\frac{1}{n}$; in other words $P_n = \{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1\}$. Let $f(x) = x$.

(a) Explicitly compute the upper and lower Riemann sums $U(P_n, f)$ and $L(P_n, f)$ (and simplify your answer).

(b) Use your answer in (a) to deduce that $f$ is integrable on $[0,1]$ directly from Theorem 21.2 (integrability criterion) (note that in the book the assertion of Theorem 21.2 is taken as the definition of integrability).

**Problem 4:** If $S$ is a non-empty subset of $\mathbb{R}$ and $f$ is a real function which is defined and bounded on $S$, we define $M(f,S) = \sup\{f(x) : x \in S\}$ and $m(f,S) = \inf\{f(x) : x \in S\}$ (a special case of this notation was used in the definition of upper and lower Riemann sums).

(a) Let $f$ and $g$ be functions defined and bounded on some non-empty $S \subseteq \mathbb{R}$. Then it is clear that the function $f + g : S \to \mathbb{R}$ is also bounded. Prove that $M(f+g,S) \le M(f,S) + M(g,S)$. **Hint:** If

$A$ is a subset of $\mathbb{R}$ and $K$ is a real number, what is a simple way to show that $\sup(A) \leq K$? If this question does not ring a bell, look at the relevant problem on the first midterm.

(b) Use (a) to prove that $U(f+g,P) \leq U(f,P) + U(g,P)$ for every partition $P$ of $[a,b]$.

(c) Now use (b) to prove that

$$\overline{\int_a^b} (f(x)+g(x))\,dx \leq \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx.$$

Note: by completely analogous argument one shows that

$$\underline{\int_a^b} (f(x)+g(x))\,dx \geq \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx.$$

(this question is not a part of the problem).

**Problem 5:** Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let $c \in (a,b)$. Let $A$ be the set of all lower Riemann sums $L(f,P)$ where $P$ is a partition of $[a,b]$, let $A_1 = \{L(f,P) : P \text{ is a partition of } [a,c]\}$ and $A_2 = \{L(f,P) : P \text{ is a partition of } [c,b]\}$. Note that by definition we have

$$\underline{\int_a^b} f(x)dx = \sup(A), \quad \underline{\int_a^c} f(x)dx = \sup(A_1) \quad \text{and} \quad \underline{\int_c^b} f(x)dx = \sup(A_2).$$

(a) Let $A'$ be the set of all lower Riemann sums $L(f,P)$ where $P$ is a partition of $[a,b]$ AND $c \in P$. Prove that $\sup(A) = \sup(A')$. **Hint:** inequality in one direction is obvious. For the other direction use Proposition 20.2 from class.

(b) Prove that $A' = A_1 + A_2$ in the sense of Homework#2.1 and deduce that $\sup(A) = \sup(A_1) + \sup(A_2)$.

(c) Deduce from (b) that $\underline{\int_a^b} f(x)dx = \underline{\int_a^c} f(x)dx + \underline{\int_c^b} f(x)dx$.

As in Problem 4, by the same argument one shows the analogue of (c) dealing with upper integrals.