Homework #1. Due Thursday, January 22nd, in class

Reading:

1. For this assignment: Section 1.2 + class notes (Lectures 1-2).
2. For next week's classes: Section 1.3.

Problems:

In this assignment, all manipulations with equalities and inequalities should be justified by explicitly referring to axioms or results proved in class or in the book. When solving 2(a), you should refer to order axioms stated in class (which are slightly different from the book) – see the end of this assignment.

Problem 1: Let $x, y \in \mathbb{R}$. Prove the following properties using only axioms and the additive cancellation law:

(a) $-(-x) = x$
(b) $-(xy) = (-x)y$

Hint: Here is how the cancellation law helps to prove identities like this. Suppose we want to prove an equality of the form $a = b$. Try to find c such that both $a + c$ and $b + c$ can be simplified. Then prove that $a + c = b + c$ directly and deduce (using the additive cancellation law) that $a = b$.

Problem 2: Let $x, y, z, w \in \mathbb{R}$.

(a) Prove that if $x > y$ and $z < 0$, then $xz < yz$ (note that in the book this is taken as one of the axioms, called the second multiplicative property).
(b) Prove that if $x > y > 0$ and $z > w > 0$, then $xz > yw$.
(c) Prove that if $x > 0$, then $x^{-1} > 0$ (use proof by contradiction)

Problem 3: Recall that in class we characterized $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers) and $\mathbb{Q}$ (rational numbers) as subsets of $\mathbb{R}$. We also proved that $n \geq 1$ for every natural number n using only the induction property of $\mathbb{N}$ (Lemma 2.1(a)) and axioms of $\mathbb{R}$. Use this fact to prove that there is no $x \in \mathbb{Z}$ such that $0 < x < 1$. (Recall that as a subset of $\mathbb{R}$, we can define $\mathbb{Z}$ as $\mathbb{Z} = \{x \in \mathbb{R} : x \in \mathbb{N} \text{ or } -x \in \mathbb{N} \text{ or } x = 0\}$).

Problem 4: Prove that it is impossible to define inequalities in $\mathbb{C}$ so that axioms (O1)-(O4) hold. **Hint:** If this was possible, then the result of Example 1.2 from the book (which asserts that $a^2 > 0$ for any $a \neq 0$) would have been true for $\mathbb{C}$ since the proof of that result uses only axioms

(O1)-(O4) and arithmetic axioms. Explain why we cannot have $a^2 > 0$ for all nonzero $a \in \mathbb{C}$.

Problem 5: Before doing this problem, read about open and closed intervals (page 13 of the book).

(a) Let $x, y \in \mathbb{R}$. Prove that $x \leq y$ if and only if $x - \varepsilon < y + \varepsilon$ for all $\varepsilon > 0$.

(b) Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there exists $z \in \mathbb{R}$ with $x < z < y$. **Hint:** You can give a simple formula for such z.

(c) Let $a, x, b \in \mathbb{R}$ with $a < x < b$. Prove that there exists $\varepsilon > 0$ such that $a < x - \varepsilon < x + \varepsilon < b$. Deduce that $(x - \varepsilon, x + \varepsilon) \subset (a, b)$ (where the symbol $\subset$ denotes proper inclusion). The importance of this result will become clear in a couple of weeks.

Hint: If you have no idea how to start on this problem, read Theorem 1.9 from the book (and make sure you understand its proof). For (c), first draw a picture and understand what the result means geometrically - this should tell you how to choose the right value of ε. Once you made a guess, prove the result formally (referring just to axioms of inequalities and their consequences proved in class/book) rather than the picture.

Problem 6: This problem introduces the concept of a metric space. Let X be a set, and let d be a function from $X \times X$ (the Cartesian product of X with itself) to $\mathbb{R}_{\geq 0}$ (non-negative real numbers) (all this means is that to every ordered pair (x, y) of elements of X we associate a non-negative real number $d(x, y)$). We say that the pair (X, d) is a *metric space* if

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality)

If (X, d) is a metric space, one should think of elements of X as points and of the number $d(x, y)$ as the distance between x and y. Properties (i)-(iii) assert that d satisfies the standard properties we would "expect" from a distance function.

Prove that the following pairs (X, d) are metric spaces:

(a) $X = \mathbb{R}$ and $d(x, y) = |y - x|$.

(b) X is any set and $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

(c) (optional) Give other example of metric spaces you can think of.

Order axioms for $\mathbb{R}$

In class, we declared the following properties to be order axioms for $\mathbb{R}$:

(O1) For all $x, y \in \mathbb{R}$ precisely one of the following is true: $x > y$ or $x = y$ or $y > x$

(O2) If $x > y$ and $y > z$, then $x > z$

(O3) If $x > 0$ and $y > 0$, then $xy > 0$

(O4) If $x > y$, then $x + z > y + z$ for all $z \in \mathbb{R}$.

Also recall that we defined order relations $<$, $\geq$ and $\leq$ in terms of $>$ by setting $x < y \iff y > x$; $x \geq y \iff x > y$ or $x = y$; $x \leq y \iff x < y$ or $x = y$.

Using these axioms, we deduced the following additional properties

(O5) If $x > z$ and $y > w$, then $x + y > z + w$

(O6) If $x > y$ and $z > 0$, then $xz > yz$.

These results along with the result of Problem 2(a) of this assignment imply that the list of order axioms from class is equivalent to the axioms in the book.