9. Characterizing sequences with convergent subsequences

Definition. Let $\{a_n\}$ be a sequence. We say that $\{a_n\}$ *diverges to* $+\infty$ and write $a_n \to +\infty$ or $\lim_{n\to\infty} a_n = +\infty$ if for any $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$.

Theorem 9.1. *Let $\{a_n\}$ be a sequence. The following are equivalent:*

(i) *$\{a_n\}$ has no convergent subsequence*
(ii) $|a_n| \to +\infty$

Before proving this theorem we establish to auxiliary results.

Claim 1. *Let $\{a_n\}$ be a sequence which diverges to $+\infty$. Then any subsequence of $\{a_n\}$ diverges to $+\infty$.*

The proof of this result is completely analogous to the proof of Lemma 7.1.

Claim 2. *Let $\{a_n\}$ be a sequence. Then $a_n \to +\infty \iff$ for any $M \in \mathbb{R}$ there are only finitely many n such that $a_n \leq M$.*

Proof. "$\Rightarrow$" Suppose that $a_n \to +\infty$ and take any $M \in \mathbb{R}$. Then by definition there exsits $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$. Hence the set of all n such that $a_n \leq M$ is a subset of $\{1, 2, \ldots, N-1\}$; in particular, this set is finite.

"$\Leftarrow$" Conversely, suppose that for any $M \in \mathbb{R}$ there are only finitely many n such that $a_n \leq M$. Now fix M and denote all values of n for which $a_n \leq M$ by $n_1, \ldots, n_t$. If we set $N = \max\{n_1, \ldots, n_t\} + 1$, then for any $n \geq N$ we must have the opposite inequality $a_n > M$. Hence by definition $a_n \to +\infty$. □

We are now ready to prove Theorem 9.1.

Proof of Theorem 9.1. "(ii)$\Rightarrow$ (i)". Suppose that $|a_n| \to +\infty$, and consider any subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Then $|a_{n_k}| \to +\infty$ (as $k \to \infty$) by Claim 1. This clearly implies that $\{a_{n_k}\}$ is not bounded, whence $\{a_{n_k}\}$ cannot converge (since convergent sequences are bounded by Lemma 7.1). Thus, we showed that any subsequence of $\{a_n\}$ does not converge.

"(i)$\Rightarrow$ (ii)". We will prove this direction by contrapositive: we shall assume that $|a_n| \not\to +\infty$ and deduce that $\{a_n\}$ has a convergent subsequence.

So suppose that $|a_n| \not\to +\infty$. Then by Claim 2, there exists $M \in \mathbb{R}$ such that there are infinitely many n satisfying $|a_n| \leq M$. Let us put those values of n in increasing order: $n_1 < n_2 < \dots$. Then $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$.

Hence, the subsequence $\{a_{n_k}\}$ is bounded, so by Bolzano-Weierstrass Theorem $\{a_{n_k}\}$ has a convergent subsequence. Since a subsequence of a subsequence is a subsequence of the original sequence, we deduce that the original sequence $\{a_n\}$ has a convergent subsequence, as desired. $\square$