Assessment exam.

This exam will test your familiarity with basic properties of sets and functions as well as basic proof techniques (contradiction and induction). The goal of this exam is to help you decide whether you should take Math 3000 Transition to Higher Mathematics prior to/concurrently with Math 3310/3354. A specific recommendation regarding Math 3000 will be made based on the results of the exam.

0. Did you already take Math 3000 or Math 3354? If yes, list the classes and when you took them.

1. Prove by contradiction that $\sqrt{12}$ is irrational.

2. Define the sequence $\{a_n\}$ by $a_1 = 4$ and $a_n = 2a_{n-1} - 3$ for all $n \geq 2$. Use mathematical induction to prove that $a_n = 2^{n-1} + 3$ for all n.

3. For each of the following functions determine whether it is injective (one-to-one) and whether it is surjective (onto). Here $\mathbb{N}$ stands for natural numbers (=positive integers), $\mathbb{R}$ for real numbers and $\mathbb{R}_{\geq 0}$ for non-negative real numbers.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$
(b) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = 2x$
(c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$
(d) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by $f(x) = x^2$

4. Let $f : X \to Y$ be a function. Given a subset A of X, let

$$f(A) = \{f(x) : x \in A\}$$

be the image of A under f (this is a subset of Y). Given a subset B of Y, let

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

be the preimage (=inverse image) of B under f, that is, $f^{-1}(B)$ is the set of all elements of X which get mapped to an element of B under f.
For each of the following statements determine whether it is true (for all possible functions) or false (for at least one function). If the statement is true, prove it; if it is false, give a specific counterexample.

(i) If B is a subset of Y, then $f(f^{-1}(B)) \subseteq B$
(ii) If B is a subset of Y, then $f(f^{-1}(B)) \supseteq B$
(iii) If A and C are subsets of X, then $f(A \cap C) = f(A) \cap f(C)$
(iv) If B and D are subsets of Y, then $f^{-1}(B \cap D) = f^{-1}(B) \cap f^{-1}(D)$