Homework #8, due on Thursday, March 31st

Reading:

1. For this assignment: Lectures 15 and 16 (March 22,24) and BT, §3.3 and 3.4

2. For next week's classes: BT, §3.4, parts of 3.5 dealing with independence (pp. 175-178) and 4.1

Problems:

Problem 1: Let $R > 0$ be a real number, and let S be the square in xy-place with vertices $(-R, -R)$, $(-R, R)$, $(R, -R)$, (R, R) (thus, S is centered at $(0, 0)$, has sides parallel to coordinate axis and side length $2R$). By a square we mean a two-dimensional figure (including the interior).

Let (X, Y) be a random point of S, with all points equally likely to occur (X and Y are the coordinates of the chosen point), and let $Z = |X| + |Y|$ (the sum of absolute values of X and Y). Find the CDF and PDF of Z.

Hint: This problem is similar to the disc problem we considered in class; the main difference is that it will take more work to describe the set S_x consisting of all points in S for which $|X| + |Y| \leq x$. To describe this set geometrically consider 4 cases: $x < 0$, $0 \leq x \leq R$, $R < x < 2R$ and $x \geq 2R$.

Problem 2: (This problem is somewhat similar Problem 6 after Chapter 3 in BT). You call a phone company to ask a question about a mysterious charge on your bill. In exactly five minutes you are connected to a company representative who will attempt to answer your question with probability 50% or will transfer you to another representative (also with probability 50%). If the representative tries to answer your question, you will still have to wait for the answer, with waiting time (in minutes) exponentially distributed with parameter $\lambda = \frac{1}{5}\ln(2)$. If you get transferred, you will again have to wait for five minutes before being connected, and the new representative will act in the same way – with probability 50% will try to answer your question, with the same distribution of waiting time (from the moment you started talking to the second representative) or transfer you again etc. (so you can get transferred any number of times). Let X be the total time of your call (in minutes).

(a) Let F_X denote the CDF of X. Prove that

$$F_X(x) = \begin{cases} 0 & \text{if } x < 5 \\ 1 - \frac{1}{2^n} - \frac{n}{2^{x/5}} & \text{if } 5n \leq x < 5(n+1) \text{ for some } n \in \mathbb{N} \end{cases}$$

(b) What is the probability that your call will last at least 30 minutes?

(c) Compute the PDF of X.

Hint for (a): Start by considering cases $5 \leq x < 10, 10 \leq x < 15$ and maybe $15 \leq x < 20$ before figuring out the general pattern. It may be easier to first find a formula for F_X for general λ and then simplify in the case $\lambda = \frac{1}{5}\ln(2)$.

Problem 3: In a city, the maximum daily temperature on April 1st is normally distributed with a mean of 64F and a standard deviation of 10F. What is the probability that the maximum temperature on April 1st in a given year will be between 60F and 70F? Express your answer in terms of the function Φ (the CDF of the standard normal random variable) and then find an approximate value using tables on 155 of BT. **Note:** A similar example is worked out in § 3.3 of BT.

Problem 4: Let X be a normal random variable, and suppose that $P(X \leq 4) = \frac{1}{3}$ and $P(X \leq 5) = \frac{2}{3}$. Find the mean of X (it can be determined precisely and almost without any calculations) and justify your answer. **Hint:** If you draw the graph of the PDF of X, what do the quantities $P(X \leq 4)$ and $P(X \leq 5)$ represent geometrically? Use these geometric interpretations and the fact that the graph is symmetric with respect to $x = \mu$ where $\mu = E[X]$.

Problem 5: Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2e^{-(x+y)} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that f is a joint PDF for some pair of random variables (X, Y)

(b) Compute f_X and f_Y, the PDFs of the random variables X and Y for which f is a joint PDF

(c) Now compute $E[X]$ and $E[Y]$.

Note: The theorem about equality of double and iterated integrals stated at the end of class on 3/24 extends naturally to improper integrals: for instance, if B is a region in xy-plane bounded by curves $x = a$, $y = u(x)$ and $y = v(x)$ where $u(x) \leq v(x)$ for all $x \geq a$, then for any continuous function f we have

$$\iint\limits_B f(x,y)dxdy = \int_a^\infty \left(\int_{u(x)}^{v(x)} f(x,y)dy \right) dx.$$