Homework #7, not due

Note: This homework is not due; however, Quiz #3 on Thu, Mar 24, will be based on this assignment.

Reading:

1. For this assignment: Lectures 13 and 14 (March 15,17), BT, §3.1 and 3.2; Durrett §5.1 and 5.2. **Warning:** In Durrett's terminology distribution functions are what BT calls *cumulative distribution functions* (we used the terminology from BT in class).
2. For next week's classes: BT, §3.3 and 3.4; Durrett §5.3 and 5.4

Problems:

Problem 1: Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} c(1-x^2) & \text{if } -1<x<1 \\ 0 & \text{otherwise,}\end{cases}$$

where c is some constant. Find the value of c for which f is a density function (of some random variable X). Make sure to justify your answer. For the value of c that you found compute the expectation and variance of the corresponding random variable.

Problem 2: Define $F:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} 0 & \text{if } x\le 0 \\ 3x^2-2x^3 & \text{if } 0<x<1 \\ 1 & \text{if } x\ge 1\end{cases}$$

Prove that F is a CDF (cumulative distribution function) of some random variable (use characterization of CDFs stated at the end of class on March 17).

Problem 3: Problem 2 at the end of Chapter 3 in BT.

Problem 4: Problem 1 at the end of Chapter 3 in BT.

Problem 5: Let X be a continuous random variable. Prove that

$$Var(aX+b)=a^2Var(X)$$

for any constants a and b. You are allowed to use the formula $E[aX+b]=aE[X]+b$ for any continuous random variable X and constants a and b. You can use without justification that if X is a continuous random variable and $a\neq 0$, then $aX+b$ is also a continuous random variable (and if $a=0$, there is almost nothing to prove). **Hint:** Computation will be shorter if you use the definition of variance, not the standard formula.

Problem 6: Suppose f is a density function, let a and b be constants, and define the function $f_{a,b}$ by $f_{a,b}(x) = a \cdot f(bx)$. Find all values of a and b for which $f_{a,b}$ is a density function. **Hint:** Consider separately the following five cases: $a < 0$, $a = 0$, ($a > 0$ and $b > 0$), ($a > 0$ and $b < 0$), ($a > 0$ and $b = 0$). Use integration by substitution to compute $\int_{-\infty}^{+\infty} f_{a,b}(x)dx$.

Problem 7: Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ \frac{2}{3} & \text{if } 1 \leq x < 2 \\ \frac{3}{4} & \text{if } 2 \leq x < 3 \text{ etc.} \end{cases}$$

Find a discrete random variable X whose CDF is equal to F (describe X by its PMF).

Problem 8: Define $F : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1+x}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Prove that F is a CDF of some random variable X and that this X is neither continuous nor discrete. **Hint:** You need to find certain property that holds for the CDF of any discrete random variable but fails for F and also certain property that holds for the CDF of any continuous random variable but fails for F (both relevant properties are listed in § 3.2 of BT).