Homework #5, to be competed before the class on Thursday, February 25th

Note: This homework is not due; however, Quiz #2 on Thu, Feb 25, will be partially based on this assignment.

Suggested reading:

1. For this assignment: BT, §2.1-2.4; Pitman §1.3, 3.1; Durrett §1.4-1.6
2. For next week's classes: BT, §2.5 and 2.7; Pitman, parts of §3.1, 3.2, 3.4; Durrett §3.4

Problems:

Problem 1: Suppose that a four-sided die is rolled twice. Let X be the number obtained on the first roll, Y the number obtained on the second roll, and consider X and Y as random variables.

(a) Find the range (set of possible values) and PMF (AKA distribution) for each of the following random variables: $X - Y$, $Y - X$, $X + Y$.
(b) Are any two of the three random variables in (a) identically distributed?

Problem 2: Fix real numbers $p, q \in (0, 1)$, let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers, and consider the function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ by $f(k) = \frac{1}{2}(p^k(1-p)+q^k(1-q))$. Explain why f is the PMF of some random variable.

Problem 3: Let X be the geometric random variable with parameter p (that is, $P(X = k) = p^k(1-p)$). Compute $Var(X)$ using the "differentiation trick" (we will compute $E(X)$ using the same method in class on Tue, Feb 23).

Problem 4: Suppose we pick 7 cards from a standard deck. What is the expected number of aces in the draw? **Hint:** Do not try to compute this by definition. Instead use certain symmetry and properties of expectation function.

Problem 5: (Durrett, 1.54) In a group of five items, two are defective. Find the distribution of N, the number of draws we need to find the first defective item. Compute the expectation and the variance of N.

Problem 6: Given a probability model Ω and an event A, the *indicator function of* A is the random variable I_A with only two possible values 0 and 1, given by $I_A = 1$ if A occurs and $I_A = 0$ if A does not occur.

(a) Prove that $E(I_A) = P(A)$ for any event A.
(b) Prove that $I_A \cdot I_B = I_{A \cap B}$ for any events A and B

(c) Give a new proof of the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ using (a), (b) and de Morgan laws (see page 5 of BT)