Homework #4. Due on Thursday, February 18th, in class

Note: This homework is not mandatory, but can be turned it. If you turn in this assignment and your score will be better than your Homework #1 score, then your Homework #1 score will be replaced by the score on this assignment.

There will be a quiz in class on Thursday, Feb 25, which will partially based on this assignment.

Note: Make sure to justify your answers. You are not expected to give super-rigorous justifications, but for instance, if your answer to a counting question is something like $3 \cdot 5 \cdot \binom{12}{2}$, you should explain what each factor in this product counts.

Suggested reading:

1. For this assignment: BT, §1.6; Pitman, §2.5, Appendix A; Durrett §2.1, parts of 2.2 and 2.4
2. For next week's classes: BT, §2.1-2.4; Pitman §1.3, 3.1; Durrett §1.4-1.6

Problems:

Practice problems from Problem BT: 1.49, 1.52, 1.53, 1.56, 1.58, 1.60

Problem 1: (Durrett 2.64). Suppose we pick 5 cards out of a deck of 52. What is the probability we get at least one card of each suit?

Problem 2: Find the probability of each of the following poker hands: (i) four of a kind, (ii) straight, (iii) three of a kind, (iv) two pair.

The wikipedia article 'List of poker hands' contains the answers, but without justifications. Note that this article uses old-fashioned notation for binomial coefficients: C_n^k instead of $\binom{n}{k}$.

Problem 3: (Durrett 2.62). A drawer contains 10 black, 8 brown, and 6 blue socks. If we pick two socks at random, what is the probability they match?

Problem 4: (Durrett 2.16) Six students, three boys and three girls, lineup in a random order for a photograph. What is the probability that the boys and girls alternate?

Problem 5: (Durrett 2.13) A person has 12 friends and will invite 7 to a party.

(a) How many choices are possible if Al and Bob are feuding and will not both go to the party?
(b) How many choices are possible if Al and Betty insist that they both go or neither one goes?

Problem 6: (Durrett 2.10) How many four-letter "words" can you make if no letter is used twice and each word must contain at least one vowel (A, E, I, O or U). By definition, a "word" is just a sequence of letters; it does not have to be a dictionary word.

Problem 7: (Durrett 2.21) How many different "words" can one obtain by rearranging the letters in each the following words: (a) money, (b) banana, (c) statistics, (d) mississippi?

Problem 8: (Durrett 2.35) A baseball pitcher throws a strike with probability 0.5 and a ball with probability 0.5. He is facing a batter who never swings at a pitch. What is the probability that he strikes out, i.e., gets three strikes before four balls?

Problem 9: Let $n, n_1, n_2, \ldots, n_k$ be non-negative integers such that $n = n_1 + n_2 + \ldots + n_k$. Prove the following formula in two different ways:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

(a) By explicitly expressing both sides in terms of factorials
(b) Using the combinatorial interpretation of binomial and multinomial coefficients

Problem 10: A *decomposition* of an integer n into k parts is a way to write $n = n_1 + n_2 + \ldots + n_k$ where each n_i is a non-negative integer (the order of terms matters!) For instance, 4 has 15 decompositions into 3 parts: $4 = 4+0+0 = 3+1+0 = 3+0+1 = 2+2+0 = 2+1+1 = 2+0+2 = 1+3+0 = 1+2+1 = 1+1+2 = 1+0+3 = 0+4+0 = 0+3+1 = 0+2+2 = 0+1+3 = 0+0+4$. Prove that the number of decompositions of n into k parts is equal to

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Hint: Suppose you have n white balls and $k-1$ black balls (where all balls of the same color are indistinguishable). What is the number of ways to line up those balls? Now think why such line-ups naturally correspond to decompositions of n into k parts.