Homework #3. Due on Thursday, February 11th, in class

Suggested reading:

1. For this assignment: BT, §1.3-1.5; Pitman, §1.4,1.5; Durrett, §1.3, 3.1-3.3
2. For next week's classes: BT, §1.6; Pitman, §1.6; Durrett §2.1

Note: Required reading assignments from BT (consisting of parts of 1.6) will be announced later.

Problems:

Problem 1: (Durrett 3.8) In a town 40% of families have a dog, 30% have a cat, and 25% of families with a dog also have a cat.

(a) What fraction of families have a dog or a cat?
(b) What is the probability a family with a cat has a dog?

Problem 2: You have two identically looking 6-sided dice. One die is fair, but the other one is loaded, so the probability of rolling 1 is $\frac{1}{2}$, and the probability of rolling any number from 2 to 6 is $\frac{1}{10}$. You pick a die at random, roll it twice and you get 1 on both rolls. What is the probability that the die you picked was fair?

Problem 3: (Pitman 1.5.4) A digital communications system consists of a transmitter and a receiver. During each short transmission interval the transmitter sends a signal which is to be interpreted as a zero, or it sends a different signal which is to be interpreted as a one. At the end of each interval the receiver makes its best guess at what was transmitted. Consider the events:

$$T_0 = \{\text{Transmitter sends } 0\}, \quad R_0 = \{\text{Receiver concludes that a 0 was sent}\},$$
$$T_1 = \{\text{Transmitter sends } 1\}, \quad R_1 = \{\text{Receiver concludes that a 1 was sent}\}.$$

Assume that $P(R_0|T_0) = 0.99, P(R_1|T_1) = 0.98$ and $P(T_1) = 0.5$. Find

(a) the probability of a transmission error given R_1
(b) the overall probability of a transmission error
(c) Repeat (a) and (b) assuming $P(T_1) = 0.8$ instead of 0.5.

Problem 4: (Pitman 1.4.10) Suppose electric power is supplied from two independent sources which work with probabilities 0.4 and 0.5, respectively. If both sources are providing power, there will be enough power with probability 1. If exactly one of them works, there will be enough power with probability 0.6. Of course, if none of them works, the probability there will be enough power is 0.

(a) What is the probability that exactly k sources work for $k = 0, 1, 2$?

(b) Compute the probability that enough power will be available.

Problem 5: Let A and B be events s.t. $P(B) = 0.6$ and $P(A \cup B) = 0.8$.

(a) Suppose that A and B are disjoint. Find $P(A)$.

(b) Suppose that A and B are independent. Find $P(A)$.

Problem 6:

(a) Suppose that events A, B and C are independent. Prove that the events $A \cup B$ and C are also independent. **Hint:** Use a suitable set identity (see page 5 of BT)

(b) Give an example showing that if A and C are independent and B and C are independent, then $A \cup B$ and C may not be independent.

(c) (optional) Show that if $\{A_1, \ldots, A_n\}$ is an independent collection of events and we replace A_i by A_i^c for some i, the new collection of events will still be independent. Applying this repeatedly, we deduce for instance that if A, B, C are independent, then the collections $\{A, B, C^c\}$, $\{A, B^c, C^c\}$ and $\{A^c, B^c, C^c\}$, are also independent.

(d) (optional) Show that if $\{A_1, \ldots, A_n\}$ is an independent collection of events and for some $i \neq j$ we replace A_i by A_j by their union or intersection, the new collection of events will still be independent. For instance, if A, B, C, D, E are independent, then $A \cup B \cup C$ and $D \cap E$ are also independent.

Problem 7: Let A, B and C be independent events and let D be the event which occurs when exactly 2 of A, B and C occur. Express $P(D)$ in terms of $P(A), P(B)$ and $P(C)$ in two different ways.

(a) Using the formula $D = (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$ and Problem 6(c).

(b) Using the formula for $P(D)$ found in HW#1, Problem 8.

Verify that the answers found in (a) and (b) are equal to each other.

Problem 8: (Durrett 1.31) Alice takes 4 courses a semester for 8 semesters. In each course she has a probability $1/2$ of getting an A. Assuming her grades in different courses are independent, what is the probability she will have at least one semester with all As? **Hint:** Problem 6(d) is relevant.

Problem 9: Suppose we draw 4 cards from a standard deck without replacement. What is the probability that the number of different suits among these cards will be equal to 3? **Hint:** Draw a tree diagram. It will be similar to the one we did in class for finding the probability that all 4 cards have different suits, but you will need more branches.