Homework #10, due on Thursday, April 14

Reading:

1. For this assignment: Lectures 17-20 (March 29,31, April 5,7) and BT, §3.5, parts of §4.1 and §4.2
2. For next week's classes: BT, §4.2, §5.1 and §5.2.

Problems:

Problem 1: Let $\lambda > 0$ be a fixed real number, and let (X, Y) be a jointly continuous pair of random variables with joint PDF $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} e^{-\lambda x - \frac{1}{\lambda} y} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove that X and Y are independent using criterion from Lecture 18. Then compute the marginal PDFs f_X and f_Y.

Problem 2: Let (X, Y) be a jointly continuous pair of random variables with joint PDF $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X + Y$. Compute PDF and CDF of Z in two different ways:

(a) Using the same method as in Problem 3 of HW#9 (this method will first give you CDF and then PDF)

(b) Using the following theorem: if X and Y are jointly continuous random variables with joint PDF $f_{X,Y}$ and $Z = X + Y$, then Z is continuous with PDF given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx \qquad (***)$$

(Thus method will first give you PDF and then CDF).

Note: The discrete analogue of formula (***) asserts the following: if X and Y are discrete random variables and $Z = X + Y$, then $p_Z(z) = \sum_{x \in R(X)} p_{X,Y}(x, z-x)$ or, equivalently, $P(Z = z) = \sum_{x \in R(X)} P(X = x, Y = z - x)$. The latter formula is clear since the event $Z = z$ is a disjoint union of events $(X = x \ \&\ Y = z - x)$ where x ranges over $R(X)$.

Geometrically, the integral in (***) represents the integral over the line $x + y = z$. Thus, to evaluate this integral in this problem you need to determine where the line $x + y = z$ intersects the region $\{(x, y) \in \mathbb{R}^2 : 0 \le x \le y\}$.

Problem 3: A student makes an appointment with a professor for 1pm, but both the student and the professor arrive at random times between 1pm and 1:15pm (uniformly distributed over this interval and independently of each other). The student will wait for the professor for 5 minutes before leaving, while the professor will wait for the same amount of time he was late by (e.g. if the professor arrives at 1:04pm, he will wait until 1:08pm). Compute the following:

(a) The probability that the professor leaves before the student arrives
(b) The probability that the student leaves before the professor arrives
(c) The probability that the meeting occurs

Problem 4: Let (X, Y) be jointly continuous with joint PDF given by Problem 1(a) in HW#9. For every $y \in (0, 1)$ compute the conditional PDF $f_{X|Y}(x|y)$ and the conditional CDF $P(X \le a|Y = y)$.

Problem 5: Consider random variables X and Y where Y is exponentially distributed with parameter $\lambda = 1$ and X is uniformly distributed on $[\frac{1}{Y+1}, \frac{1}{Y}]$. Compute the joint PDF $f_{X,Y}$ and the marginal PDF f_X.

Problem 6: Read at least the beginning of 4.1 before doing this problem. Let X be a random variable that is uniformly distributed on $[-1, 1]$, and let $Y = \sqrt{|X|}$ and $Z = -\ln|X|$. Compute the PDF and CDF of Y and the PDF and CDF of Z.

Problem 7: Let X_1 and X_2 be the numbers on two independent rolls of a fair die. Let $X = X_1 - X_2$ and $Y = X_1 + X_2$. Prove that X and Y are uncorrelated, but not independent.