Homework #1. Due on Friday, January 29th, by 3pm in my mailbox

Abstractions:

(a) BT: "Introduction to Probability" by Bertsekas and Tsitsiklis
(b) P: "Probability" by Pitman
(c) D: "Elementary Probability for Applications" by Durrett
(d) Symbol $\sim$ next to a problem means that the assigned problem is very similar to one of the book problems
(e) Symbol + next to a problem means that the assigned problem is a book problem with extra part(s) added

Note: There is an excellent web supplement to BT which, in particular, contains a complete solution manual.

`http://www.athenasc.com/probbook.html`

Suggested reading:

1. For this assignment: BT, §1.1-1.2; P, §1.1,1.3; B, §1.1,1.2
2. For next week's classes: BT, §1.1-1.4; P, §1.1,1.3-1.5; B, §1.1,1.2

Note: Required reading assignments from BT (consisting of parts of the above sections) will be announced later.

Problems:

Problem 1: (P 1.1.7+) Suppose we roll a pair of fair (6-sided) dice, so the sample space Ω is the set $\{(i, j)\}$ where i and j are integers between 1 and 6, with all outcomes equally likely. Compute the probability of each of the following events. In (a) and (b) also give an explicit description of the event as a subset of Ω:

(a) the maximum of the two numbers rolled is less than or equal to 2
(b) the maximum of the two numbers rolled is less than or equal to 3
(c) the maximum of the two numbers rolled is exactly equal to 3
(d) Repeat (b) and (c) replacing 3 by x, where x ranges from 1 to 6 (do this separately for each x)
(e) Denote by $P(x)$ the probability that the maximum number is exactly x. What should $P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$ equal to? First give a conceptual argument (without any calculations) and then check that it is consistent with your answers in (d).

Problem 2: Suppose we have a probability model where the sample space $\Omega = [0, 1]$, the closed interval from 0 to 1 (one explicit such model has been considered at the end of Lecture 1). Is it possible that

(a) $P([a, b]) = a - b$ for every $0 \leq a \leq b \leq 1$?
(b) $P([a, b]) = (b - a)^2$ for every $0 \leq a \leq b \leq 1$?
(c) $P([a, b]) = \sqrt{b} - \sqrt{a}$ for every $0 \leq a \leq b \leq 1$?

Note that (a),(b) and (c) are separate questions. Right now we do not have the tools to give a (rigorous) argument for a 'yes' answer, but you can give a rigorous 'no' answer by showing that at least one of probability axioms (or one of their consequences – see pp. 9 and 14 in BT) fails.

Problem 3: BT 1.5

Problem 4: ($\sim$ BT 1.6) A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is

(a) even
(b) divisible by 3

Problem 5: ($\sim$ BT 1.7) A four-sided die is rolled repeatedly, until the first time (if ever) that we obtain the same number on two consecutive rolls.

(a) What is the sample space for this experiment?
(b) Let $P(x)$ denote the probability that the number of tosses is at most x (where x is a positive integer). Find an explicit formula for $P(x)$.

Problem 6: ($\sim$ BT 1.8) Solve the problem analogous to BT 1.8 where you play 4 opponents instead of 3 and you need to win 3 (not 2) consecutive games. **Optional:** (harder) What if you play 4 opponents but only need to win 2 consecutive games?

Problem 7: (P 1.3.2) Write down the expression in the set notation for each of the following events:

(a) the event which occurs if exactly one of the events A and B occurs
(b) the event which occurs if none of the events A, B and C occurs
(c) the events obtained by replacing "none" in (b) by "exactly one", "exactly two" and "all three"

Problem 8: (P 1.3.10) Events A, B and C are defined in a sample space. Find expressions for the following probabilities in terms of $P(A), P(B), P(C)$, $P(A \cap B), P(A \cap C), P(B \cap C)$ and $P(A \cap B \cap C)$

(a) the probability that exactly two of A, B and C occur
(b) the probability that exactly one of A, B and C occurs
(c) the probability that none of A, B and C occurs

For each part first write down the event in question using set notation and then use properties of probability laws to compute the given probability.