Homework #6. Due on Friday, Match 16th by 1pm in TA's mailbox

Reading:

1. For this assignment: sections 3.3 and 3.4 of the BOOK (throughout this assignment BOOK refers to the book 'A discrete transition to advanced mathematics').
2. For next week's classes: sections 4.1 and 4.2.

Practice problems from the BOOK: From 3.3: 3, 8, 10, 18. From 3.4: 1, 4, 8, 11.

Problems to hand in:

1. Each of the following statements is an identically true implication (as proved in class) depending on some free variables with values in $\mathbb{Z}$:

(i) Let $a, b, c \in \mathbb{Z}$. If $c \mid a$ and $c \mid b$, then $c \mid (a + b)$ (here a, b and c are free variables)

(ii) Let $a, b \in \mathbb{Z}$. If $a \mid b$, then $a \mid bd$ for all $d \in \mathbb{Z}$ (here a and b are free variables and d is a bound variable)

(iii) Let $a, b \in \mathbb{Z}$, and let $p \in \mathbb{P}$ where $\mathbb{P}$ is the set of all prime numbers. If $p \mid ab$, then $p \mid a$ or $p \mid b$ (here p, a, b are free variables)

For each of the implications (i), (ii) and (iii) formulate the converse statement, determine whether the converse is identically true or generally false and prove your answer.

2. Let $n \in \mathbb{N}$ with $n \geq 2$, and write $n = p_1^{a_1} \ldots p_k^{a_k}$ where $p_1, \ldots, p_k$ are distinct primes and each $a_i \in \mathbb{N}$. Prove that n is a perfect square $\iff$ each a_i is even. Recall that we proved the "$\Leftarrow$" direction in Lecture 12, so you only need to prove the "$\Rightarrow$" direction. **Warning/Hint:** If you did not refer to the uniqueness factorization, your argument is likely incomplete.

3. Prove Corollary 3.3.4 from the book which can be restated as follows: Let $m, n \in \mathbb{N}$ with $n \geq 2$, and write $n = p_1^{a_1} \ldots p_k^{a_k}$ where $p_1, \ldots, p_k$ are distinct primes and each $a_i \in \mathbb{N}$. Then $m \mid n \iff$ there exist $b_1, \ldots, b_k \in \mathbb{Z}_{\geq 0}$ such that $m = p_1^{b_1} \ldots p_k^{b_k}$ and $b_i \leq a_i$ for each i. **Note:** The goal of this problem is to give a completely formal proof (a sketch is already given in the book).

4. Problem 2 in Section 3.3. **Hint:** Suppose we are given $m, n, d \in \mathbb{Z}$ and we want to prove that $d = gcd(m, n)$. By definition of gcd this amounts to verifying two properties:

(i) $d \mid m$ and $d \mid n$

(ii) If c is any integer such that $c \mid m$ and $c \mid n$, then $c \leq d$.

5. Given an integer n and a prime p, define $\mathrm{ord}_p(n)$ to be the multiplicity with which p occurs in the prime factorization of n. If p does not occur in the prime factorization of n at all, set $\mathrm{ord}_p(n) = 0$. For instance,

$$\mathrm{ord}_p(45) = \begin{cases} 2 & \text{if } p = 3 \\ 1 & \text{if } p = 5 \\ 0 & \text{if } p \neq 3, 5. \end{cases}$$

Note that the results of Problems 2 and 3 can be restated as follows in terms of the ord_p function:

(2) Let $n \in \mathbb{N}$. Then n is a perfect square if and only if $\mathrm{ord}_p(n)$ is even for each prime p

(3) Let $m, n \in \mathbb{N}$. Then $m \mid n$ if and only if $\mathrm{ord}_p(m) \leq \mathrm{ord}_p(n)$ for each prime p.

Prove the following properties:

(a) $\mathrm{ord}_p(mn) = \mathrm{ord}_p(m) + \mathrm{ord}_p(n)$ for each prime p and for all $m, n \in \mathbb{N}$

(b) $\mathrm{ord}_p(m + n) \geq \min(\mathrm{ord}_p(m), \mathrm{ord}_p(n))$ for each prime p and for all $m, n \in \mathbb{N}$. Give an example showing that the inequality may be strict.

(c) Let $m, n \in \mathbb{N}$. Then m and n are relatively prime $\iff$ for each prime p we have $\mathrm{ord}_p(n) = 0$ or $\mathrm{ord}_p(m) = 0$.

6. Problem 17 in 3.3.

7. Problem 3 in 3.4.