**Homework #5. Due on Friday, February 23rd by 1pm in TA's mailbox**

**Reading:**

1. For this assignment: sections 3.1 and 3.2 of the BOOK (throughout this assignment BOOK refers to the book 'A discrete transition to advanced mathematics').
2. For next week's classes: section 3.3 and the beginning of 3.4.

**Practice problems from the BOOK:** From 3.1: 4, 6, 9, 17, 20. From 3.2: 5, 7, 11.

**Problems to hand in:**

**1.** Problem 12 in 2.2 from the BOOK.

**2.** Let $n \in \mathbb{N}$, and let $\{A_1, \ldots, A_n\}$ be a collection of $n$ finite sets such that any three distinct sets in the collection have empty intersection, that is, $A_i \cap A_j \cap A_k = \emptyset$ whenever $i, j, k$ are distinct (we do NOT assume that the sets $\{A_i\}$ are disjoint). Use induction on $n$ to prove that

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|.$$

**Hint:** In order to complete the induction step you would probably want to use two formulas for the cardinality of unions established earlier in class.

**3.** Locate the mistake in the following "proof" by strong induction.
Let $S_n$ be the statement "$2^{n-1} = 1$." We claim that $S_n$ is true for all $n \in \mathbb{N}$.
*Base case:* $S_1$ is true since $2^{1-1} = 2^0 = 1$.
*Strong induction step:* Now fix $n \in \mathbb{N}$, and suppose that $S_k$ is true for all $1 \le k \le n$. We need to show that $S_{n+1}$ is true.
By definition of $S_{n+1}$, we need to show that $2^n = 1$. Using exponent laws, we can write

$$2^n = 2^{(2n-2)-(n-2)} = \frac{2^{2n-2}}{2^{n-2}} = \frac{(2^{n-1})^2}{2^{n-2}}. \qquad (***)$$

By induction hypothesis $S_n$ and $S_{n-1}$ are both true, so $2^{n-1} = 1$ and $2^{n-2} = 1$. Thus, from the equation (***) we get $2^n = \frac{1^2}{1} = 1$, so $S_{n+1}$ is also true.

**4.** Solve Problem 2 in 3.1 directly using definition of divisibility.

**5.** Problem 1(a)(d) in 3.2.

**6.** Let $a, b \in \mathbb{Z}$ with $(a, b) \neq (0, 0)$, and let $d = gcd(a, b)$. Let

$$d\mathbb{Z} = \{dk \mid k \in \mathbb{Z}\}$$

be the set of all multiples of $d$, and let

$$L_{a,b} = \{am + bn \mid m, n \in \mathbb{Z}\}$$

be the set of all integer linear combinations of $a$ and $b$. Prove that $L_{a,b} = d\mathbb{Z}$ by showing that each of these two sets is contained in the other.

**7.** Problem 7 in 3.1.

**8.** Problems 21 and 22 in 3.1.