Homework #4. Due on Friday, February 16th by 1pm in TA's mailbox

Reading:

1. For this assignment: sections 2.1 and 2.2 of the BOOK (throughout this assignment BOOK refers to the book 'A discrete transition to advanced mathematics').
2. For next week's classes: sections 3.1 and 3.2. My current plan is to spend about half a class on induction next Tuesday and then move to Chapter 3. Section 2.3 (pigeonhole principle) will most likely be covered later, together with topics from Chapter 4.

Practice problems from the BOOK: From 2.1: 2, 5, 6, 8, 9, 22. From 2.2: 2, 4, 9, 14, **15**.

Problems to hand in:

1. In both parts of this problem A, B and C are sets.

(a) Prove that $A \times (B \cap C) = (A \times B) \cap A \times C$
(b) Assume that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. Prove that $B \subseteq C$. Explain why the conclusion may be false if $A = \emptyset$ and clearly state where you use that $A \neq \emptyset$ in your proof.

If you are not sure how to write down the proofs formally, consult section 5.4 of the 'Mathematical reasoning' book (see the url below) for plenty of worked out examples of similar kind.

`https://www.tedsundstrom.com/mathreasoning`

2. Prove that there are no integers a and b such that $a^2 = 4b + 2$ (equivalently, if we divide the square of any integer by 4 with remainder, the remainder is never equal to 2).

3. Let x and y be real numbers, at least one of which is irrational. Prove that at least one of the numbers $x + y$ and $x - y$ is irrational.

4. Problem 10 in 2.1 from the BOOK.

5. Problem 19 in 2.1 from the BOOK.

6. Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

7. In Lecture 7 we proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$. Moreover, it is shown that such a_n and b_n

satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1+\sqrt{3})^n = c_n + d_n\sqrt{3}$.

(c) (bonus) Find a simple formula relating c_n and d_n (similar to the one in (a)) and prove it.

8. Use induction to solve Problem 7 from 2.2 in the BOOK.