Homework #3. Due on Friday, February 9th by 1pm in TA's mailbox

Reading:

1. For this assignment: sections 1.5 and 1.6 of the BOOK (throughout this assignment BOOK refers to the book 'A discrete transition to advanced mathematics').
2. For next week's classes: sections 2.1 (general proof techniques) and 2.2 (mathematical induction). My current plan is to talk about the basic material from 2.1 on Tuesday, talk about induction on Thursday and introduce other techniques from 2.1 at a later point.

Practice problems from the BOOK: From 1.5: 1, 2(c)(f), 3, 4(a)(d), 6. From 1.6: 4, 6, 7, 10, 11.

Problems to hand in:

1. Let $\{B_i \mid i \in I\}$ be an indexed collection of subsets of a set U, and let A be a subset of U. Prove the generalized distributivity laws (a) and (b) as instructed below:

(a) $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I}(A \cap B_i)$
(b) $A \cup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I}(A \cup B_i)$

For (a) give a proof similar to the proof of de Morgan laws given in Lecture 4 (or § 1.5 of the book). Then deduce (b) from (a) and de Morgan laws.

2. Problem 2(a)(b)(d)(e) in 1.5 from the BOOK. Make sure to explain why the statement is true/false.

3. Problem 4(b)(c)(e) in 1.5 from the BOOK. Make sure to explain why the statement is true/false.

4. Express the following statements in symbolic form using quantifiers:

(a) Distinct real numbers have distinct cubes.
(b) There exists the smallest natural number.

5. Problem 5 in 1.6 from the BOOK.

6. Problem 7 in 1.6 from the BOOK.

7. Let P, Q, R and S be statements. If U is any statement obtained from P, Q, R and S using logical connectives $\wedge$, $\vee$, $\sim$ and $\Longrightarrow$, the *complexity of* U is the number of logical connectives that it involves. For instance, the statement $(P \vee Q) \Longrightarrow (P \vee S)$ has complexity 3 (we used $\Longrightarrow$ once and $\vee$ twice).

In each of the following examples find a statement V which is equivalent to the given statement U such that either V has smaller complexity than U or V has the same complexity as U, but involves fewer occurrences of $\implies$. Prove that your V is indeed equivalent to U.

(a) U is $P \implies (Q \implies R)$
(b) U is $(P \implies Q) \vee (P \implies R)$
(c) U is $P \wedge (\sim Q) \wedge (\sim R) \wedge (\sim S)$

Hint: There are several ways to figure out the answer. First, you can rewrite U using English sentences (and think of a different, but equivalent formulation). Second, you can try to use the laws we already established. Finally, you can write down the truth table for U, although this will not automatically tell you how to find V.

8.

(a) Let P, Q and R be arbitrary statements. Prove that the following statements are equivalent: $(P \implies R) \vee (Q \implies R)$ and $(P \wedge Q) \implies R$.

- (b) Locate a logical mistake in the following argument:

  Consider the following statements.
  - $P : n$ is divisible by 2
  - $Q : n$ is divisible by 3
  - $R : n$ is divisible by 6

In each statement we assume that $n \in \mathbb{Z}$ and consider n as a free variable. The implication $P \implies R$ is false since there exist integers which are divisible by 2, but not divisible by 6, e.g. $n = 2$. Likewise, the implication $Q \implies R$ is also false since $n = 3$ is divisible by 3, but not divisible by 6. Hence $(P \implies R) \vee (Q \implies R)$ is also false (being the conjunction of two false statements).

On the other hand, the implication $(P \wedge Q) \implies R$ asserts that if an integer n is divisible by both 2 and 3, then it is divisible by 6. This is a true implication (as we will prove in a few weeks).

Thus, in the above example $(P \implies R) \vee (Q \implies R)$ is false while $(P \wedge Q) \implies R$ is true. This contradicts (a).