**Homework #11. Due on Friday, April 27th by 1pm in TA's mailbox**

**Reading:**

1. For this assignment: sections 6.3 and 6.4 of the BOOK.
2. For next week's classes: section 6.4.

**Practice problems from the BOOK:** From 6.3: 6, 9, 10, 13; from 6.4: 3, 4.

**Problems to hand in:**

**1.** Problem 1(a)(b) from Section 6.3. Recall that we proved 1(c) in Lecture 22. Give a detailed argument.

**2.** Problem 2 from Section 6.4. Note: the problem should say $f^{-1}(\{i\})$ is countable, not $|f^{-1}(\{i\})|$ is countable. **Hint:** Let $A_i = f^{-1}(\{i\})$. How is the collection of sets $\{A_i\}$ related to $A$?

**3.** Let $A_1, \ldots, A_n$ be countable sets. Prove that the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is countable. **Hint:** Argue by induction on $n$. In the case $n = 1$ there is nothing to prove, and in the case $n = 2$ the result is Corollary 6.3.10 in the book (make sure to either prove it yourself or read the proof in the book). If you are not sure how to set up the induction step, go over Lecture 8 notes.

**4.** In Lecture 24 (Tue, April 24) we will prove that $\mathbb{Q}$ is countable using Theorem 23.5 from class (a countable union of countable sets is countable). Give another proof of countability of $\mathbb{Q}$ by constructing a surjective map $g : A \times B \to \mathbb{Q}$ for certain countable sets $A$ and $B$ (and using suitable theorems).

**5.** Let $A$ be an uncountable set and $B$ a countable subset of $A$.

(a) Prove that $A \setminus B$ is uncountable.

(b) Prove that $A$ and $A \setminus B$ have the same cardinality.

**Hint for (b):** Since $A \setminus B$ is infinite by (a), by Theorem 6.3.5 from the book we can choose a countably infinite subset $C$ of $A \setminus B$. Use things proved in class to show that the identity map $f : (A \setminus B) \setminus C \to (A \setminus B) \setminus C$ can be extended to a bijection from $A \setminus B$ and $A$. Draw a picture!

**6.** A real number $\alpha$ is called algebraic if $\alpha$ is a root of a (nonzero) polynomial with **integer** coefficients, that is, if there exist integers $c_0, \ldots, c_n$, not all 0 such that $\sum\limits_{k=0}^{n} c_k \alpha^k = 0$. Note that all rational numbers are algebraic (if $\alpha = \frac{p}{q}$, then $\alpha$ is a root of the polynomial $qx - p$), but many irrational numbers are algebraic as well (e.g. $\sqrt{2}$ is algebraic as $\sqrt{2}$ is a root of $x^2 - 2$). The goal of this problem is to prove that the set of all algebraic numbers is countable.

(a) For a fixed integer $n \geq 0$, let $Z_n$ be the set of all polynomials of degree at most $n$ with integer coefficients, that is, $Z_n$ is the set of all polynomials of the form $\sum_{k=0}^{n} c_k x^k$ with each $c_i \in \mathbb{Z}$. Prove that each $Z_n$ is countable. **Hint:** Construct a bijection between $Z_n$ and a Cartesian product of finitely many countable sets and use the result of Problem 2.

(b) Now use (a) (and a suitable theorem) to show that the set of polynomials with integer coefficients (of arbitrary degree) is countable.

(c) Finally use (b) and the fact that every polynomial has finitely many roots to show that the set of all algebraic numbers is countable.

**7.** Problem 3 from Section 6.3. For parts (a) and (b) construct an explicit bijection between the given sets; one way to solve (c) is to use a suitable theorem from Section 6.4.

**8.** (BONUS) Let $A$ be any infinite set and $B$ any non-empty countable set. Prove that $|A \times B| = |A|$. You can use the following fact without proof: any infinite set can be written as a disjoint union of countably infinite sets.